

HW 4

Q1.

1. BEC transition make the macroscopic numbers of particles be ground states.
BEC doesn't happen for fermions because of Pauli exclusion law.

2. $f(x) = \frac{1}{4}x^4 + \frac{a}{2}x^2$

$$\frac{\partial f}{\partial x} = x^3 + ax = 0 \rightarrow x = 0, \pm\sqrt{-a}, \quad \frac{\partial^2 f}{\partial x^2} = 3x^2 + a$$

Case $a > 0$: real soln is only $x = 0$ and it is global minimum from $\frac{\partial^2 f(0)}{\partial x^2} > 0$ ψ

Case $a < 0$: There are 3 solutions $x = 0, \pm\sqrt{-a}$

$$\frac{\partial^2 f}{\partial x^2}(\pm\sqrt{-a}) > 0 \quad \text{and} \quad \frac{\partial^2 f(0)}{\partial x^2} < 0$$

so $x = \pm\sqrt{-a}$ are global minimum and it is broken sym case ψ

3. There was $U(1)$ symmetry and total number of particles are conserved.

There is Nambu-Goldstone mode $\omega_q = cq$ with $c = \sqrt{\frac{\hbar^2}{m}} = \sqrt{\frac{\hbar^2}{m}}$

Q2.

$$G_{ij}(\tau) = -\langle T_\tau \psi_i(\tau) \psi_j^\dagger(0) \rangle \quad \text{when } \psi_i(\tau) = e^{H\tau} \psi_i e^{-H\tau}$$

$$G_{ij}(\tau) = -\frac{1}{2} \int \mathcal{D}[\bar{\psi}, \psi] \psi_i(\tau) \bar{\psi}_j(0) e^{-S[\bar{\psi}, \psi]} \quad \text{when } S = \sum_i \int_0^\beta d\tau \bar{\psi}_i (\partial_\tau + \epsilon_i - \mu) \psi_i$$

1. $\Delta\tau = \beta/N$. $e^{-\beta H} = (e^{-\Delta\tau H})^N$

$$1 = \int d[\psi_k \bar{\psi}_k] e^{-\bar{\psi}_k \psi_k} |\psi_k\rangle \langle \psi_k| \quad (k=0, \dots, N-1)$$

$$G_{ij}(\tau) = -\text{Tr} [e^{-\beta H} T_\tau (\psi_i(\tau) \psi_j^\dagger(0))]$$

$$= \int \prod_{k=0}^{N-1} d[\psi_k, \bar{\psi}_k] e^{-\sum_k \bar{\psi}_k \psi_k} \langle \psi_0 | e^{-\Delta\tau H} | \psi_1 \rangle \langle \psi_1 | \dots \rangle \dots \langle \psi_{N-1} | e^{-\Delta\tau H} | \psi_0 \rangle \psi_i(\tau) \psi_j^\dagger(0)$$

$$= \int \mathcal{D}[\bar{\psi}, \psi] \psi_{i,m} \bar{\psi}_{j,0} e^{-S[\bar{\psi}, \psi]} \quad \text{when } \tau = \Delta\tau \cdot m$$

continuum $\rightarrow \int \mathcal{D}[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(0) e^{-S}$

Case $\tau > 0$: $T_\tau (\psi_i(\tau) \psi_j^\dagger(0)) = \psi_i(\tau) \psi_j^\dagger(0)$ and it does match to $\psi_{i,m}$ naturally appear as LFT of $\bar{\psi}_{j,0}$ from above formulation

Case $\tau < 0$: $T_\tau (\psi_i(\tau) \psi_j^\dagger(0)) = \psi_j^\dagger(0) \psi_i(\tau)$

$\psi_i(\tau) = \psi_i(\tau + \beta)$ so order swapped and match

2. $\psi_i(\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \psi(i\omega_n)$ where $\omega_n = \begin{cases} 2\pi n/\beta & \text{boson} \\ (2n+1)\pi/\beta & \text{fermion} \end{cases}$

$$S = \sum_i \int_0^\beta d\tau \bar{\psi}_i (\partial_\tau + \epsilon_i - \mu) \psi_i$$

$$= \sum_i \frac{1}{\beta} \sum_n \bar{\psi}_i(i\omega_n) (i\omega_n + \epsilon_i - \mu) \psi_i(i\omega_n)$$

Two point GF $\langle \psi_i(i\omega_n) \bar{\psi}_j(i\omega_m) \rangle = \beta \delta_{ij} \delta_{nm} G_i^0(i\omega_n) = \beta \delta_{ij} \delta_{nm} \frac{1}{i\omega_n + \epsilon_i - \mu}$

$$\rightarrow \langle \psi_i(\tau) \bar{\psi}_j(0) \rangle = \frac{1}{\beta} \sum_n e^{i\omega_n \tau} \delta_{ij} \frac{1}{-i\omega_n + \epsilon_i - \mu}$$

$$\text{Thus, } G_{ij}(\tau) = \frac{1}{\beta} \sum_n \bar{e}^{i\omega_n \tau} \delta_{ij} \frac{1}{i\omega_n - \epsilon_i + \mu}$$

$$\frac{1}{\beta} \sum_n f(i\omega_n) = \oint_c \frac{dz}{2\pi i} n_{B,F}(z) f(z)$$

$$G_{ij}(\tau) = \pm \oint_c \frac{dz}{2\pi i} \delta_{ij} n_{B,F}(z) \cdot \frac{\bar{e}^{i z \tau}}{z - \epsilon_i + \mu} = \begin{cases} -\delta_{ij} \bar{e}^{-(\epsilon_i - \mu)\tau} [1 + n_B(\epsilon_i - \mu)] & \text{Boson} \\ -\delta_{ij} \bar{e}^{-(\epsilon_i - \mu)\tau} [1 - n_F(\epsilon_i - \mu)] & \text{fermion} \end{cases} \quad (0 < \tau < \beta)$$

$$3. \lim_{\tau \rightarrow 0^-} G_{ii}(\tau) = \lim_{\tau \rightarrow 0^-} \frac{1}{\beta} \sum_n \frac{\bar{e}^{i\omega_n \tau}}{i\omega_n - \epsilon_i + \mu} = \frac{1}{\beta} \sum_n \frac{1}{i\omega_n - \epsilon_i + \mu} = n_{B/F}(\epsilon_i - \mu)$$

$$\begin{aligned} 4. \quad Z[J, \bar{J}] &= \int \mathcal{D}[\bar{\psi}, \psi] \exp \left\{ -\sum_i \int_0^\beta d\tau \bar{\psi}_i (\partial_\tau + \epsilon_i - \mu) \psi_i - \sum_i \int_0^\beta d\tau (\bar{J}_i \psi_i + \bar{\psi}_i J_i) \right\} \\ &= \int \mathcal{D}[\bar{\psi}, \psi] \exp \left\{ \frac{-1}{\beta} \sum_{i,n} (\bar{\psi}_i(i\omega_n) (-i\omega_n + \epsilon_i - \mu) \psi_i(i\omega_n) + \bar{J}_i(i\omega_n) \psi_i(i\omega_n) + \bar{\psi}_i(i\omega_n) J_i(i\omega_n)) \right\} \\ &= Z_0 \exp \left[\frac{1}{\beta} \sum_{i,n} \bar{J}_i(i\omega_n) \frac{1}{-i\omega_n + \epsilon_i - \mu} J_i(i\omega_n) \right] \end{aligned}$$

$$\begin{aligned} G_{ij}(i\omega_n) &= -\langle T_\tau \psi_i(i\omega_n) \bar{\psi}_j(i\omega_n) \rangle = \frac{-1}{Z[0,0]} \frac{\delta^2 Z[J, \bar{J}]}{\delta \bar{J}_i(i\omega_n) \delta J_j(i\omega_n)} \Big|_{\bar{J}=\bar{J}=0} \\ &= \delta_{ij} \frac{1}{i\omega_n - \epsilon_i + \mu} \end{aligned}$$

Q3.

$$\hat{H} = -\frac{1}{2} \sum_{ij} J_{ij} \hat{S}_i \cdot \hat{S}_j \quad (\forall i, j, J_{ij} > 0)$$

$$\hbar, \text{ s.t. } |\hbar| = 1,$$

$$\text{ground state } |0_n\rangle = \bigotimes_{i=1}^N |i, n\rangle, \quad n \cdot \hat{S}_i |i, n\rangle = -\frac{1}{2} |i, n\rangle$$

1. $\hbar$ in spherical coordinate $\hbar = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$

$$|i, \hbar\rangle = \cos\frac{\theta}{2} |\downarrow\rangle + e^{i\phi} \sin\frac{\theta}{2} |\uparrow\rangle$$

$$|i, \hbar_1\rangle = |\downarrow\rangle, \quad |i, \hbar_2\rangle = \cos\frac{\theta}{2} |\downarrow\rangle + e^{i\phi} \sin\frac{\theta}{2} |\uparrow\rangle$$

$$|0_{\hbar_1}\rangle = \bigotimes_{i=1}^N |i, \hbar_1\rangle, \quad |0_{\hbar_2}\rangle = \bigotimes_{i=1}^N |i, \hbar_2\rangle$$

$$\rightarrow |\langle 0_{\hbar_1} | 0_{\hbar_2} \rangle| = \prod_{i=1}^N |\langle i, \hbar_1 | i, \hbar_2 \rangle| = \left(\cos\frac{\theta}{2}\right)^N$$

$$\forall \theta \neq 0, \cos\frac{\theta}{2} < 1 \quad \text{Thus, } |\langle 0_{\hbar_1} | 0_{\hbar_2} \rangle| \rightarrow 0 \text{ for } N \rightarrow \infty.$$

2. $\hbar = \mathbb{E}_2$.

$$S_i^\pm = S_i^x \pm i S_i^y \rightarrow S_i^x = \frac{1}{2} (S_i^+ + S_i^-), \quad S_i^y = \frac{1}{2i} (S_i^+ - S_i^-)$$

$$\hat{H} = -\frac{1}{2} \sum_{ij} J_{ij} \left[S_i^x S_j^x + \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) \right]$$

$$(S_i^+ - S_i^-) (S_j^- - S_j^+) = S_i^+ S_j^- + S_j^+ S_i^- - S_i^+ S_j^+ - S_j^+ S_i^-$$

$$S_i^+ S_j^- S_j^+ S_j^- + \frac{1}{4} = S_i^x S_j^x + \frac{1}{2} (S_i^+ S_j^- + S_j^+ S_i^-)$$

$$\hat{H} = -\frac{1}{2} \sum_{ij} J_{ij} \left[-\frac{1}{2} (S_i^+ - S_i^-) (S_j^- - S_j^+) + S_i^+ S_j^- S_j^+ S_j^- + \frac{1}{4} \right]$$

3. $|i\rangle = S_i^+ |0_{\mathbb{E}_2}\rangle$

$$\hat{H}_{1p} = \frac{1}{4} \sum_{ij} J_{ij} (S_i^+ - S_j^+) (S_i^- - S_j^-) = \frac{1}{2} \sum_j J_{ij} (|i\rangle - |j\rangle)$$

$$|k\rangle = \sum_j e^{ikx_j} |j\rangle$$

$$\hat{H}_{1p} |k\rangle = \sum_j e^{ikx_j} \hat{H}_{1p} |j\rangle = \sum_j e^{ikx_j} \frac{1}{2} \sum_l J_{jl} (|j\rangle - |l\rangle)$$

$$= \frac{1}{2} \sum_j |j\rangle \sum_l J_{jl} e^{ikx_j} - \frac{1}{2} \sum_l |l\rangle \sum_j J_{jl} e^{ikx_j} = \frac{1}{2} (J_0 - J_k) |k\rangle$$

$$J_k = \sum_n J e^{-ik \cdot r_n} = J (e^{-ik_x a} + e^{ik_x a} + e^{-ik_y a} + e^{ik_y a}) = 2J (\cos k_x a + \cos k_y a)$$

$$J_0 = \sum_n J = 4J$$

$$E_k = \frac{1}{2} (J_0 - J_k) = J [2 - (\cos k_x a + \cos k_y a)]$$